Contact between linearly elastic bodies: the Signorini problem

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Problem setting

- **The problem**
- **The model**

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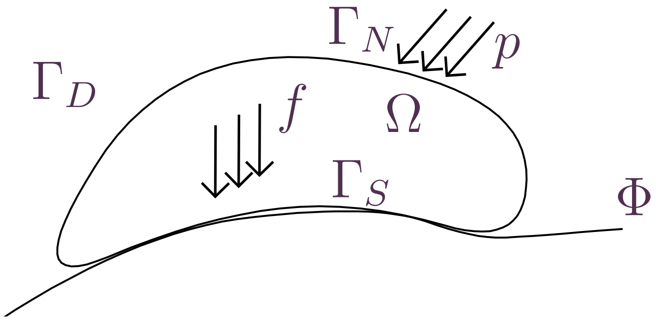

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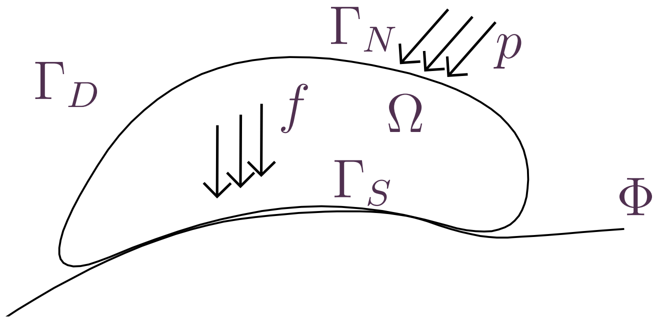

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  - Anisotropic, homogeneous, **linearly elastic** body in stationary equilibrium on a **rigid foundation**. No friction, no thermodynamics, no fancy stuff.

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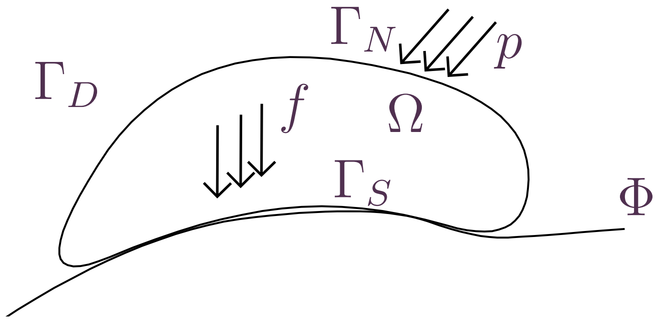

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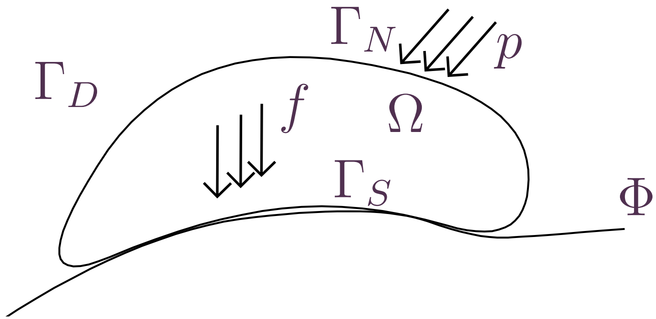

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  - Anisotropic, homogeneous, **linearly elastic** body in stationary equilibrium on a **rigid foundation**. No friction, no thermodynamics, no fancy stuff.
  - Small displacements: $|u| \ll 1$, $|\nabla u| \ll 1$.
  - Stress tensor given by Hooke's law.

$$\sigma_{ij}(u) = a_{ijkl}\,\varepsilon_{kl}(u) = \frac{1}{2}\,a_{ijkl}\,(u_{k,l} + u_{l,k}).$$

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  Stress-free reference configuration: $\Omega \subset \mathbb{R}^3$ open, bounded, with $C^{1,1}$ boundary $\Gamma = \partial\Omega$ split in open smooth subsets Γ_D, Γ_N, Γ_S. Γ_D possibly empty.

  Given some data f, p, g find the **displacements** u such that

$$\begin{cases} -\operatorname{div}\sigma(u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma_D, \\ \sigma\nu = p & \text{on} \quad \Gamma_N, \end{cases}$$

  and on Γ_S:

$$\begin{cases} u\nu - g \leqslant 0, \\ (u\nu - g)(\nu\sigma\nu) = 0, \\ \nu\sigma\nu \leqslant 0 \\ \nu\sigma\tau = 0. \end{cases}$$

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Boundary conditions

- **Neumann**
- **Dirichlet**
- **Signorini (contact)**

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  Tractions given on $\Gamma_N \subset \Gamma$.
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  - The case $|\Gamma_D| > 0$.
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    Possible incompatibility at the boundary $\Rightarrow$ need for $H_{00}^{1/2}(\Sigma)$.
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  Kinematical non-penetration and conditions on normal stresses at $\Gamma_S \subseteq \mathrm{Int}\,(\Gamma \backslash \Gamma_D)$.

  $\Rightarrow$ **Free boundary** problem.

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- **Stresses at the boundary**

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- **Stresses at the boundary**
  - No friction $\Rightarrow$ no tangential stresses.
  - Stresses are normal at contact points.
  - There is normal stress *iff* there is contact.
  - Together, on Γ_S we have:

$$\nu_i\,\sigma_{ij}\,\tau_j = 0$$

$$\begin{cases} u_i\,\nu_i - g & = & 0, \\ \sigma_{ij}\,\nu_i\,\nu_j & < & 0, \end{cases} \text{ or } \begin{cases} u_i\,\nu_i - g & < & 0, \\ \sigma_{ij}\,\nu_i\,\nu_j & = & 0. \end{cases}$$

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Assume $|\Gamma_D| > 0$.

Let: $\Sigma = \Gamma \setminus \overline{\Gamma}_D$, $f \in \boldsymbol{L}^2(\Omega)$, $p \in \boldsymbol{H}^{-1/2}(\Gamma_N)$, $g \in H^{1/2}_{00}(\Sigma)$.

Define: $V := \{v \in \boldsymbol{H}^1(\Omega) \colon v = 0 \text{ in } \Gamma_D\}$ and $\gamma_{\Sigma_\nu} \colon V \to H^{1/2}_{00}(\Sigma)$ (**normal trace**).

Admissible set: $K := \{v \in V \colon \gamma_{\Sigma_\nu}(v)\,\nu - g \leqslant 0 \text{ on } \Gamma_S \text{ in } H^{1/2}_{00}(\Sigma)\}$.

Energy functional: $\boldsymbol{I(v) := \frac{1}{2}\,a(v,v) - F(v), \quad v \in K}$.

Find $u \in K$ such that $\boldsymbol{I(u) = \min_{v \in K} I(v)}$.

Where

$$a(u,v) := \int_\Omega a_{ijkl}\,\varepsilon_{kl}(u)\,\varepsilon_{ij}(v)\,\mathrm{d}x = \int_\Omega \sigma_{ij}(u)\,\varepsilon_{ij}(v)\,\mathrm{d}x, \quad \varepsilon(v) := \frac{1}{2}(\nabla^\top v + \nabla v)$$

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- **Not so easy**.

  Equivalence with the original PDEs. Definition of traces and Green's formulas.

Assume $|\boldsymbol{\Gamma_D}| = \mathbf{0}$.

Let $f \in \boldsymbol{L}^2(\Omega)$, $p \in \boldsymbol{H}^{-1/2}(\Gamma_N)$, $g \in H^{1/2}(\Gamma)$.

Take now

$$K := \{v \in V : \gamma_{\Gamma_\nu}(v)\,\nu - g \leqslant 0 \text{ on } \Gamma_S \text{ in the sense of } H^{1/2}(\Gamma)\}.$$

Find $u \in K$ such that for every $v \in K$:

$$a(u, v - u) \geqslant F(v - u) \qquad (1)$$

Equivalence (thanks to Gâteaux differentiability of the energy)

$$u \text{ is a minimizer} \Longleftrightarrow u \text{ solves } (1).$$

(with proper choice of K, etc.).

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Find $u \in K$ such that for every $v \in K$:

$$a(u, v - u) \geqslant F(v - u) \tag{2}$$

Equivalence (thanks to Gâteaux differentiability of the energy)

$$u \text{ is a minimizer} \Longleftrightarrow u \text{ solves } (2).$$

(with proper choice of K, etc.).

Assume $|\boldsymbol{\Gamma_D}| = \mathbf{0}$.

Let $f \in \boldsymbol{L}^2(\Omega), p \in \boldsymbol{H}^{-1/2}(\Gamma_N), g \in H^{1/2}(\Gamma)$.

Take now

$$K := \{v \in V : \gamma_{\Gamma_\nu}(v)\,\nu - g \leqslant 0 \text{ on } \Gamma_S \text{ in the sense of } H^{1/2}(\Gamma)\}.$$

Find $u \in K$ such that for every $v \in K$:

$$a(u, v - u) \geqslant F(v - u) \tag{3}$$

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$$u \text{ is a minimizer} \Longleftrightarrow u \text{ solves } (3).$$

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Find $u \in K$ such that for every $v \in K$:

$$a(u, v - u) \geqslant F(v - u) \tag{4}$$

Equivalence (thanks to Gâteaux differentiability of the energy)

$$u \text{ is a minimizer} \iff u \text{ solves } (4).$$

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Assume $|\boldsymbol{\Gamma_D}| = \mathbf{0}$.

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$$K := \{v \in V\colon \gamma_{\Gamma_\nu}(v)\,\nu - g \leqslant 0 \text{ on } \Gamma_S \text{ in the sense of } H^{1/2}(\Gamma)\}.$$

Find $u \in K$ such that for every $v \in K$:

$$a(u, v-u) \geqslant F(v-u) \tag{5}$$

Equivalence (thanks to Gâteaux differentiability of the energy)

$$u \text{ is a minimizer} \Longleftrightarrow u \text{ solves } (5).$$

(with proper choice of K, etc.).

Assume $|\boldsymbol{\Gamma_D}| = \mathbf{0}$.

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Take now

$$K := \{v \in V : \gamma_{\Gamma_\nu}(v)\,\nu - g \leqslant 0 \text{ on } \Gamma_S \text{ in the sense of } H^{1/2}(\Gamma)\}.$$

Find $u \in K$ such that for every $v \in K$:

$$a(u, v - u) \geqslant F(v - u) \tag{6}$$

Equivalence (thanks to Gâteaux differentiability of the energy)

$$u \text{ is a minimizer} \Longleftrightarrow u \text{ solves } (6).$$

(with proper choice of K, etc.).

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- **(Alternative?) proof idea:** Consider solutions u_ρ in bounded sub-cones $K_\rho = \{v \in K: \|v\| \leqslant \rho\}$. Prove that there must exist one with $\|u_\rho\| < \rho$ using the compatibility condition.

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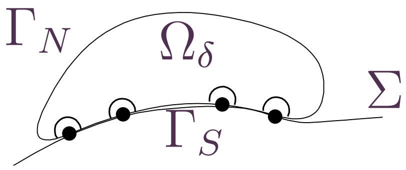
$$u \in \boldsymbol{H}^2(\Omega_\delta) \ \textit{for every} \ \delta > 0$$

and the "classical" boundary conditions are fulfilled in $\Gamma_D \cup \Gamma_N \cup \Gamma_S$. *In particular for* $N = 3$, *the solution is in* $\boldsymbol{C}^{0,1/2}(\overline{\Omega}_\delta) \cap \boldsymbol{W}^{1,6}(\Omega_\delta)$.



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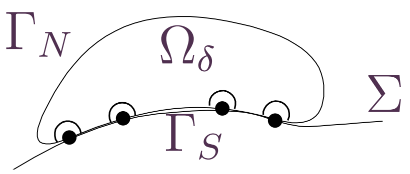

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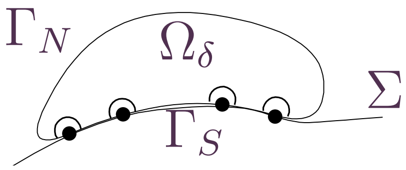

- **Higher regularity**

(Two bodies) *Boieri, Gastaldi, Kinderlehrer, 1987*.

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- **$u \in V$ solution and Ω of class $C^{1,1}$**

Thanks to $\gamma : V \twoheadrightarrow H_{00}^{1/2}(\Gamma_S)^N$ *surjective*, there exists a *unique* operator

$$\pi : \{\sigma \in L^2(\Omega)^{N\times N}_{\text{sym}} : \operatorname{div} \sigma \in L^2(\Omega)^N\} \longrightarrow H_{00}^{-1/2}(\Gamma_S)^N$$

such that

$$\int_\Omega \operatorname{div} \sigma(u) \cdot v + \int_\Omega \sigma(u) : \varepsilon(v) = \langle \pi(\sigma(u)), v \rangle_{H_{00}^{-1/2}}$$

and

$$\pi(\sigma(u)) = \sigma(u)\,\nu$$

for smooth functions.

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- **Planar estimates**

- **The scalar problem**

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Let $g \in H^{1/2}(\Gamma)$, $p \in H^{-1/2}(\Gamma)$, $f \in L^2(\Omega)$ such that

$$F(1) := (f, 1)_{L^2} + \langle p, 1 \rangle_{H^{-1/2}(\Gamma)} < 0.$$

Find $u \in K_g := \{v \in H^1(\Omega) : v \geqslant g \text{ on } \Gamma\}$ such that

$$a(u, v-u) \geqslant F(v-u) \text{ for all } v \in K_g$$

where $a(u,v) := (\nabla u, \nabla v)_{L^2}$. Equivalently, find $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u &= f & \text{in } \Omega, \\ u &\geqslant g & \text{in } H^{1/2}(\Gamma), \\ \partial_\nu u &\geqslant p & \text{in } H^{-1/2}(\Gamma), \\ (u-g)\,(\partial_\nu u - p) &= 0 & \text{a.e. } \Gamma. \end{cases}$$

The coincidence set for the scalar problem (II)

$$\Gamma_g := \Gamma \setminus \{x \in \Gamma : u(x) > g(x) \text{ in the sense of } H^{1/2}(\Gamma)\},$$

or formally: $\Gamma_g := \{x \in \Gamma : u(x) = g(x)\}$.

Theorem. (Díaz / d.B.) *Let $\Omega \Subset \mathbb{R}^N$ with C^3 boundary and let $u \in H^1(\Omega) \cap L^\infty(\Omega)$ solve the scalar problem. Let $\|u\|_\infty \leqslant M$. Assume there exists a set $\Gamma_\delta \subset \Gamma$ where the following holds:*

$$p - \partial_\nu v_0 \leqslant -\delta := \frac{-1}{M},$$

where $v_0 \in H^2(\Omega)$ is the solution of the Dirichlet problem

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- **Basic tool** for the proof

The coincidence set for the scalar problem (III)

- **Basic tool** for the proof

  Comparison principle:

Lemma. *Let* $f_1 \leqslant f_2 \in L^2(\Omega)$ *and let* $u_1, u_2 \in H^1(\Omega)$ *be scalar functions defined on an open, bounded and connected set* $\Omega \subset \mathbb{R}^N$ *with Lipschitz boundary* $\Gamma = \partial\Omega$ *such that*

$$\begin{cases} -\Delta u_i &= & f_i & a.e. \ \Omega, \\ \partial_\nu u_1 &\leqslant & \partial_\nu u_2 & over \ \Gamma_\nu \ in \ H^{-1/2}(\Gamma), \\ u_1 &\leqslant & u_2 & over \ \Gamma_u \ in \ H^{1/2}(\Gamma), \end{cases}$$

where $\Gamma_\nu \uplus \overline{\Gamma}_u = \Gamma$ *and* $\mathrm{Int}\,\Gamma_u \neq \emptyset$. *Then* $u_1 \leqslant u_2$ *almost everywhere in* Ω.

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- **Application**

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- **Application**

  Build local supersolution using intrinsic distance.

Plan for numerics:

1. Two body problem.
2. Saddle point formulation.
3. Mortar method.
4. Examples.

- Basically the same problem.

The two body problem

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Saddle point formulation

- Lagrange multiplier space: $Q := \boldsymbol{H}_{00}^{-1/2}(\Gamma_S^a)$.
- For any $\mu \in Q$ define μ_ν, μ_τ *normal and tangential components*. Let

$$Q_+ := \{\mu \in Q \colon \mu_\tau = 0,\, \mu_\nu \leqslant 0 \text{ on } \Gamma_S^a \text{ in } H_{00}^{-1/2}(\Gamma_S^a)\}.$$

- Let $v_\nu^\iota = \gamma^0_{\Gamma_S^\iota,\nu}(v^\iota)$. For every $v \in V = V^a \times V^b$, $\mu \in Q$ define the bilinear form

$$b(v,\mu) := \langle \mu_\nu, v_\nu^a \rangle + \langle \mu_\nu, v_\nu^b \rangle.$$

- Find $(u,\lambda) \in V \times Q_+$ such that

$$\begin{cases} a(u,v) + b(v,\lambda) & = & F(v), & v \in V, \\ b(u,\mu-\lambda) & \leqslant & g((\mu-\lambda)_\nu), & \mu \in Q_+. \end{cases}$$

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- **Discretization** of function spaces $\hat{V}$, $\hat{Q}$... and conditions $\hat{g}$, $\hat{\mu}^{a,b}$...
- **Matching condition** instead of pointwise matching $\hat{v}^a \leqslant \hat{v}^b$:

$$\int_{\hat{\Gamma}} (\hat{v}^a - \hat{v}^b)\, \hat{g}^a \leqslant 0 \text{ for every } \hat{g}^a \in \Lambda_\delta^a = (\dots)$$

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The mortar method

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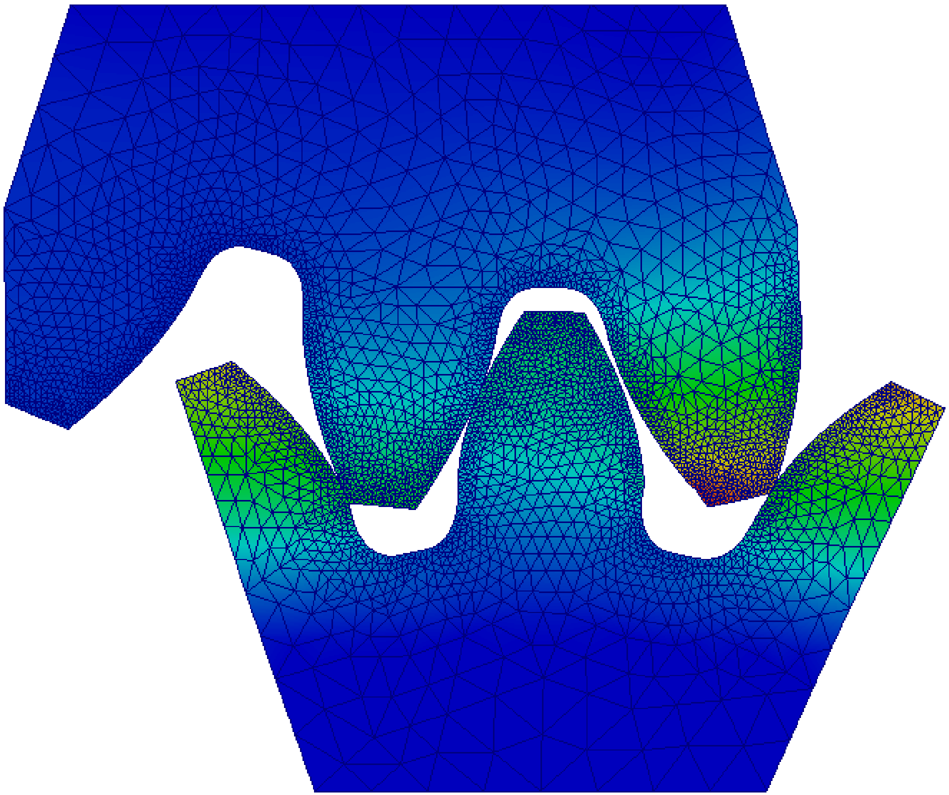
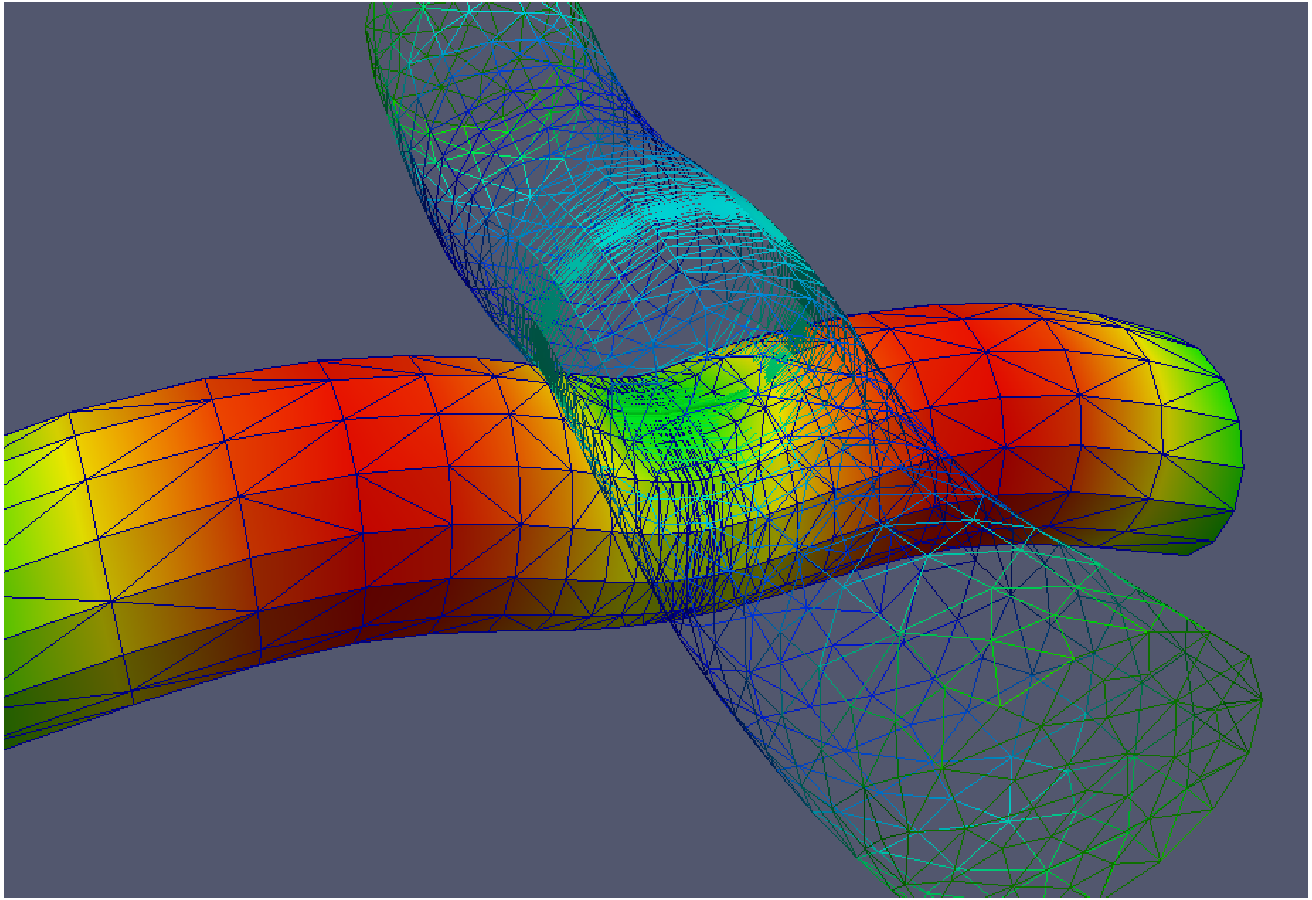


Horizontal displacements of two touching cogs pressed together.



Forces well above the tolerance of the linear model still **seem** to provide reasonable results. Color represents vertical displacement.

Related problems

- Stefan's problem
- Dam problem
- Subsonic flow
- Magnetohydrodynamics
- Plasticity...